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Hacène Djellout, Yacouba Samoura

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HACÈNE DJELLOUT AND YACOUBA SAMOURA

Abstract. In this note, we consider the large and moderate deviation principle of the estimators of the integrated covariance of two-dimensional diffusion processes when they are observed only at discrete times in a synchronous manner. The proof is extremely simple. It is essentially an application of the contraction principle for the results given in the case of the volatility [4].

AMS 2000 subject classifications: 60F10, 62J05, 60J05.

1. Motivation and context

Given a filtred probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, let $(X_{1,t}, X_{2,t})$ be a two dimensional diffusion process given by

$$\begin{cases} dX_{1,t} = u_{1,t}(X_{1,t})dt + \sigma_{1,t}dB_{1,t} \\ dX_{2,t} = u_{2,t}(X_{2,t})dt + \sigma_{2,t}dB_{2,t} \end{cases} \tag{1.1}$$

where $((B_{1,t}, B_{2,t}), t \geq 0)$ is a two-dimensional Gaussian process with independent increments, zero mean and covariance matrix

$$\begin{pmatrix} t & \int_0^t \rho_s ds \\ \int_0^t \rho_s ds & t \end{pmatrix} \qquad \forall t \geq 0.$$

In (1.1), (u_1, u_2) is a progressively measurable process (possibly unknown). In what follows, we restrict our attention to the case when σ_1, σ_2 and ρ are deterministic functions; the functions σ_i, $i = 1, 2$ take positive values while ρ takes values in the interval $[-1, 1]$. Note that the marginal processes B_1 and B_2 are Brownian motions (BM). Moreover, we can define a process B_t^* such that $(B_{1,t}, B_t^*)_{t\geq 0}$ is a two-dimensional BM and $dB_{2,t} = \rho_t dB_{1,t} + \sqrt{1-\rho_t^2}dB_t^*$ for every $t \geq 0$.

In this note, the parameter of interest is the (deterministic) covariance of X_1 and X_2

$$\langle X_1, X_2\rangle_t = \int_0^t \sigma_{1,t}\sigma_{2,t}\rho_t dt. \tag{1.2}$$

In finance,$\langle X_1, X_2\rangle.$ is the integrated covariance (over $[0, 1]$) of the logarithmic prices X_1 and X_2 of two securities. It is an essential quantity to be measured for risk management purposes. The covariance for multiple price processes is of great interest in many financial applications. The naive estimator is the realized covariance, which is the analogue of realized variance for a single process.

Typically $X_{1,t}$ and $X_{2,t}$ are not observed in continuous time but we have only discrete time observations. Given discrete equally spaced observation $(X_{1,t_k^n}, X_{2,t_k^n}, k = 1, \cdots, n)$ in

Date: October 11, 2013.

Key words and phrases. Deviation inequalities, Moderate deviation principle, Large deviation principle, Diffusion, Discrete-time observation, Quadratic variation, Realised volatility.

the interval $[0,1]$ (with $t_k = k/n$), a commonly used approach to estimate is to take the sum of cross products

$$\mathbf{C}_t^n := \sum_{k=1}^{[nt]} \left(X_{1,t_k^n} - X_{1,t_{k-1}^n} \right) \left(X_{2,t_k^n} - X_{2,t_{k-1}^n} \right), \tag{1.3}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

When the drift is known, we can also consider the following estimator

$$\overline{\mathbf{C}}_t^n := \sum_{k=1}^{[nt]} \left(X_{1,t_k^n} - X_{1,t_{k-1}^n} - \int_{t_{k-1}^n}^{t_k^n} u_{1,t}(X_{1,t})dt \right) \left(X_{2,t_k^n} - X_{2,t_{k-1}^n} - \int_{t_{k-1}^n}^{t_k^n} u_{2,t}(X_{2,t})dt \right)$$

In the unidimensional case and in the case that X have non-jump, this question has been well investigated see [4] for relevant references. In [4] and recently in [7], the authors obtained the large and moderate deviations for the realized volatility. The results of [4] are extended to jump-diffusion processes. Mancini [8] established the large deviation result for the threshold estimator for the constant volatility. Hui [6] derived a moderate deviation result for the threshold estimator for the quadratic variational process.

In the bivariate case, Hayashi and Yoshida [5] considered the problem of estimating the covariation of two diffusion processes under a non-synchronous sampling scheme. They proposed an alternative estimator and they investigated the asymptotic distributions. In [2], the authors complement the results in [5] by establishing a second-order asymptotic expansion for the distribution of the estimator in a fairly general setup, including random sampling schemes and (possibly random) drift terms. Several further works have been realized when data on two securities are observed non-synchronously, see also [1]. Here we do not consider the asynchronous case. In the bivariate case we also mention the work of Mancini and Gobbi [9] which deal with the problem of distinguishing the Brownian covariation from the co-jumps using a discrete set of observations.

The purpose of this note is to furnish some further estimations about the estimator (1.3), refining the already known central limit theorem. More precisely, we are interested in the estimations of

$$\mathbb{P}\left(\frac{\sqrt{n}}{b_n} \left(\mathbf{C}_t^n - \int_0^t \sigma_{1,t}\sigma_{2,t}\rho_t dt \right) \in A \right),$$

where A is a given domain of deviation, $(b_n)_{n>0}$ is some sequence denoting the scale of the deviation. When $b_n = 1$, this is exactly the estimation of the central limit theorem. When $b_n = \sqrt{n}$, it becomes the *large deviations*. And when $1 \ll b_n \ll \sqrt{n}$, this is the so called *moderate deviations*. The main problem studied in this paper is the large and moderate deviations estimations of the estimator. In this bivariate case things are not complicated.

We refer to Dembo and Zeitouni [3], for an exposition of the general theory of large deviation and limit ourself to the statement of the some basic definitions. Let $\{\mu_T, T > 0\}$ be a family of probability on a topological space $(S, \mathcal{S})$ where $\mathcal{S}$ is a σ-algebra on S and $v(T)$ a non-negative function on $[1, \infty)$, such that $\lim_{T\to\infty} v(T) = +\infty$. A function $I : S \to [0, \infty]$ is said to be a rate function if it is lower semicontinuous and it is said to be a good rate function if its level set $\{x \in S : I(x) \le a\}$ is compact for all $a \ge 0$. $\{\mu_T\}$ is said to satisfy a large deviation principle (LDP) with speed $v(T)$ and rate function $I(x)$ if for any set $A \in \mathcal{S}$

$$-\inf_{x\in A^\circ} I(x) \le \lim_{T\to\infty} \begin{pmatrix} \inf \\ \sup \end{pmatrix} \frac{1}{v(T)} \log \mu_T(A) \le -\inf_{x\in \bar{A}} I(x).$$

where $A^0, \bar{A}$ are the interior and the closure of A respectively.

This paper is organized as follows. In the next section we present the main results of this paper. They are established in the last section.

2. Main results

Our first result is about the LDP of $\mathbb{P}(\mathbf{C}_1^n \in \cdot)$, with time $t = 1$ fixed.

Proposition 2.1. *Let $(X_{1,t}, X_{2,t})$ be given by (1.1).*

(1) For every $\lambda \in \mathbb{R}$

$$\begin{aligned}\Lambda_n(\lambda) &:= \frac{1}{n}\log \mathbb{E}(\exp(\lambda n \overline{\mathbf{C}}_1^n)) \\ &\le \Lambda(\lambda) := \begin{cases} \displaystyle\int_0^1 -\frac{1}{2}\log(1-\lambda\sigma_{1,t}\sigma_{2,t}(1+\rho_t)) - \frac{1}{2}\log(1+\lambda\sigma_{1,t}\sigma_{2,t}(1-\rho_t))dt \\ \qquad\qquad \textit{if } -\dfrac{1}{||\sigma_1\sigma_2(1-\rho)||} \le \lambda \le \dfrac{1}{||\sigma_1\sigma_2(1+\rho)||} \\ +\infty, \qquad \textit{otherwise.}\end{cases}\end{aligned}$$

and

$$\lim_{n\to\infty} \Lambda_n(\lambda) = \Lambda(\lambda).$$

(2) Assume that $\sigma_{1,\cdot}\sigma_{2,\cdot}(1 \pm \rho_\cdot) \in L^\infty([0,1], dt)$ and $u_{l,\cdot}(\cdot) \in L^\infty(dt \otimes \mathbb{P})$, for $l = 1, 2$. Then $\mathbb{P}(\mathbf{C}_1^n \in \cdot)$ satisfies the LDP on $\mathbb{R}$ with speed n and with the good rate function given by the Legendre transformation of Λ, that is

$$\Lambda^*(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - \Lambda(\lambda)\}. \tag{2.1}$$

We now extend Proposition 2.1 to the process-level large deviations of $\mathbb{P}(\mathbf{C}_\cdot^n \in \cdot)$, which is interesting from the viewpoint of the non-parametric statistics.

Let $\mathbb{D}_b([0,1])$ be the real right-continuous-left-limit and bounded variation functions γ. The space $\mathbb{D}_b([0,1])$ of γ, identified in the usual way as the space of bounded measures $d\gamma$ on $[0,1]$, with $d\gamma[0,t] = \gamma(t)$ and $d\gamma(0) = \gamma(0)$, will be equipped with the weak convergence topology and the σ-field $\mathcal{B}^s$ generated by the coordinate $\{\gamma(t), 0 \le t \le 1\}$. We denote by $\dot{\gamma}(t)dt$ and $d\gamma^\perp$ respectively the absolute continuous part and the singular part of the measure $d\gamma$ associated with $\gamma \in \mathbb{D}_b[0,1]$ w.r.t. the Lebesgue measure dt. The signed measure γ has a unique decomposition into a difference $\gamma = \gamma_+ - \gamma_-$ of two positive measures γ_+ and γ_-. In the paper, we denote by P^* the function

$$P^*(x) = \begin{cases} \dfrac{1}{2}(x - 1 - \log x) & \text{if} \quad x > 0 \\ \\ \qquad +\infty & \text{if} \quad x \le 0,\end{cases} \tag{2.2}$$

which is the Legendre transformation of P given by

$$P(\lambda) = \begin{cases} -\dfrac{1}{2}\log(1-2\lambda) & \text{if } \lambda < \dfrac{1}{2} \\ +\infty, & \text{otherwise.}\end{cases} \tag{2.3}$$

Theorem 2.2. *Let $(X_{1,t}, X_{2,t})$ be given by (1.1). Assume that $\sigma_{1,\cdot}\sigma_{2,\cdot}(1\pm\rho_\cdot) \in L^\infty([0,1], dt)$ and $u_{l,\cdot}(\cdot) = u_l(\cdot,\cdot) \in L^\infty(dt \otimes \mathbb{P})$, for $l = 1, 2$. Then*

(1) $\mathbb{P}(\mathbf{C}^n_\cdot \in \cdot)$ *satisfies the LDP on* $\mathbb{D}_b([0,1])$ *w.r.t. the weak convergence topology, with speed* n *and with some inf-compact convex rate function* $J(\gamma)$.

(2) *If moreover* $t \to \sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t)$ *is continuous and strictly positive on* $[0,1]$, *then*

$$J(\gamma) = J_+^{abs}(\gamma_+ + \beta) + J_-^{abs}(\gamma_- + \beta) + J_+^{\perp}(\gamma_+) + J_-^{\perp}(\gamma_-), \tag{2.4}$$

where β *is absolutely continuous with respect to the Lebesgue measure and given by*

$$\dot{\beta}(t) = \frac{\sigma_{1,t}\sigma_{2,t}(1-\rho_t^2) - (\dot{\gamma_+}(t) + \dot{\gamma_-}(t))}{2} + \frac{\sqrt{[\sigma_{1,t}\sigma_{2,t}(1-\rho_t^2) - (\dot{\gamma_+}(t) + \dot{\gamma_-}(t))]^2 + (\dot{\gamma_+}(t) + \dot{\gamma_-}(t))\sigma_{1,t}\sigma_{2,t}(1-\rho_t^2)}}{2},$$

and

$$J_\pm^{\perp}(\gamma) = \int_0^1 \frac{1}{\sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t)} d\gamma^{\perp},$$

and

$$J_\pm^{abs}(\gamma) = \int_0^1 P^* \left(\frac{2\dot{\gamma}(t)}{\sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t)} \right) dt,$$

where P^* *is given in (2.2).*

We discuss now the moderate deviations principle. To this purpose, let $(b_n)_{n \geq 1}$ be a sequence of positive numbers such that

$$b_n \to \infty \qquad \text{and} \quad \frac{b_n}{\sqrt{n}} \to 0 \qquad \text{as} \quad n \to \infty.$$

Let $\mathbb{D}_0[0,1]$ be the Banach space of real right-continuous-left-limit functions γ on $[0,1]$ with $\gamma(0) = 0$, equipped with the uniform sup norm and the σ-field $\mathcal{B}^s$ generated by the coordinate $\{\gamma(t), 0 \leq t \leq 1\}$.

Theorem 2.3. *Given* $(X_{1,t}, X_{2,t})$ *by (1.1) with* $u_{l,\cdot}(\cdot) = u_l(\cdot,\cdot) \in L^\infty(dt \otimes \mathbb{P})$, *for* $l = 1,2$. *Assume that* $\sigma_{1,\cdot}\sigma_{2,\cdot}(1 \pm \rho_\cdot) \in L^2([0,1], dt)$ *and*

$$\sqrt{n} b_n \max_{1 \leq k \leq n} \int_{(k-1)/n}^{k/n} \sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t) dt \longrightarrow 0. \tag{2.5}$$

Then $\mathbb{P}\left(\frac{\sqrt{n}}{b_n} (\mathbf{C}^n_\cdot - \langle X_1, X_2 \rangle_\cdot) \in \cdot \right)$ *satisfies the LDP on* $\mathbb{D}_0([0,1])$ *with speed* b_n^2 *and with the good rate function* J_m *given by*

$$J_m(\gamma) = \begin{cases} \displaystyle\int_0^1 \frac{\dot{\gamma}(t)^2}{2\sigma_{1,t}^2\sigma_{2,t}^2(1+\rho_t^2)} 1_{[t:\sigma_{1,t}\sigma_{2,t}>0]} dt & \text{if } d\gamma \ll \sigma_{1,t}\sigma_{2,t}\sqrt{1+\rho_t^2} dt \\ +\infty & \text{otherwise} \end{cases} \tag{2.6}$$

Remark 2.4. *In particular,* $\mathbb{P}\left(\frac{\sqrt{n}}{b_n} (\mathbf{C}^n_1 - \langle X_1, X_2 \rangle_1) \in \cdot \right)$ *satisfies the LDP on* $\mathbb{R}$ *with speed* b_n^2 *and with the rate function given by*

$$I_m(x) = \frac{x^2}{2\int_0^1 \sigma_{1,s}^2\sigma_{2,s}^2(1+\rho_s^2) ds}, \quad \forall x \in \mathbb{R}.$$

Remark 2.5. *If for some $p > 2$,*

$$\sigma_{1,\cdot}\sigma_{2,\cdot}(1 \pm \rho_\cdot) \in L^p([0,1], dt) \quad \text{and} \quad b_n = O\left(n^{\frac{1}{2}-\frac{1}{p}}\right),$$

we obtain (2.5).

Remark 2.6. *Theorem 2.2 and 2.3 continue to hold under the linear growth condition of the drift u_l $(l = 1, 2)$ rather than the boundedness. More precisely assume that*

$$|u_{l,s}(x) - u_{l,t}(y)| \le \alpha_l \left[1 + |x - y| + \eta_l(|s - t|)(|x| + |y|)\right], \ \forall s, t \in [0,1], x, y \in \mathbb{R},$$

where $\eta_l : [0, +\infty) \to [0, +\infty)$ is a continuous nondecreasing function with $\eta_l(0) = 0$ and $\alpha_l > 0$ is a constant. Then the LDP of Theorem 2.2 and 2.3 continue to hold for $\mathbb{P}(\tilde{\mathbf{C}}^n_\cdot \in \cdot)$, where $\tilde{\mathbf{C}}^n_\cdot$ is given by

$$\tilde{\mathbf{C}}^n_t := \sum_{k=1}^{[nt]} \left(X_{1,t_k^n} - X_{1,t_{k-1}^n} - u_{1,t_{k-1}^n}(X_{1,t_{k-1}^n})(t_k^n - t_{k-1}^n)\right)$$
$$\left(X_{2,t_k^n} - X_{2,t_{k-1}^n} - u_{2,t_{k-1}^n}(X_{2,t_{k-1}^n})(t_k^n - t_{k-1}^n)\right).$$

We introduce the following function:

$$\Lambda^*_\pm(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda_\pm(\lambda)\}, \tag{2.7}$$

which is the Legendre transformation of $\Lambda_\pm$ given by

$$\Lambda_\pm(\lambda) := \int_0^1 P\left(\pm \frac{\lambda \sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t)}{2}\right) dt. \tag{2.8}$$

And we denote by

$$\alpha_{\pm,t} = \frac{1}{2}\int_0^t \sigma_{1,s}\sigma_{2,s}(1 \pm \rho_s) ds. \tag{2.9}$$

An easy application of deviation inequalities given in Proposition 1.5 in [4] gives

Proposition 2.7. *We have for every $n \ge 1$ and $r > 0$,*

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,1]} [\overline{\mathbf{C}}^n_t - \mathbb{E}\overline{\mathbf{C}}^n_t] \ge r\right) &\le \exp\left(-n\Lambda^*_+(\alpha_+ + \frac{r}{2})\right) + \exp\left(-n\Lambda^*_-(\alpha_- - \frac{r}{2})\right) \\
&\le \exp\left(-\frac{n}{2}\left[\frac{r}{\|\sigma_1\sigma_2(1+\rho)\|_\infty} - \log\left(1 + \frac{r}{\|\sigma_1\sigma_2(1+\rho)\|_\infty}\right)\right]\right) \\
&\quad + \exp\left(-n\frac{r^2}{4\int_0^1 [\sigma_1\sigma_2(1-\rho)]^2 dt}\right),
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left(\inf_{t\in[0,1]} [\overline{\mathbf{C}}^n_t - \mathbb{E}\overline{\mathbf{C}}^n_t] \le -r\right) &\le \exp\left(-n\Lambda^*_+(\alpha_+ - \frac{r}{2})\right) + \exp\left(-n\Lambda^*_-(\alpha_- + \frac{r}{2})\right) \\
&\le \exp\left(-n\frac{r^2}{4\int_0^1 [\sigma_1\sigma_2(1+\rho)]^2 dt}\right) \\
&\quad + \exp\left(-\frac{n}{2}\left[\frac{r}{\|\sigma_1\sigma_2(1-\rho)\|_\infty} - \log\left(1 + \frac{r}{\|\sigma_1\sigma_2(1-\rho)\|_\infty}\right)\right]\right),
\end{aligned}$$

where $\Lambda^*_\pm$ and $\alpha_\pm$ are given in (2.7) and (2.9) respectively.

3. Proof

In this section, we will give some hints for the proof of the main results. We have the following

$$\overline{\mathbf{C}}_t^n = \sum_{k=1}^{[nt]} \left(\int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s} dB_{1,s} \right) \left(\int_{t_{k-1}^n}^{t_k^n} \sigma_{2,s} dB_{2,s} \right) = \sum_{k=1}^{[nt]} \sqrt{a_k} \sqrt{a'_k} \xi_k \xi'_k,$$

where

$$\xi_k = \frac{\int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s} dB_{1,s}}{\sqrt{a_k}} \quad \xi'_k = \frac{\int_{t_{k-1}^n}^{t_k^n} \sigma_{2,s} dB_{2,s}}{\sqrt{a'_k}} \quad \text{with} \quad a_k = \int_{t_{k-1}^n}^{t_k^n} \sigma_{1,t}^2 dt \quad a'_k = \int_{t_{k-1}^n}^{t_k^n} \sigma_{2,t}^2 dt.$$

Obviously $((\xi_k, \xi'_k))_{k=1,\cdots,n}$ are independent centered Gaussian random vector with covariance

$$\frac{1}{\sqrt{a_k}\sqrt{a'_k}} \begin{pmatrix} \sqrt{a_k}\sqrt{a'_k} & \int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}\sigma_{2,s}\rho_s ds \\ \int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}\sigma_{2,s}\rho_s ds & \sqrt{a_k}\sqrt{a'_k} \end{pmatrix}.$$

Let us introduce the following notation:

$$\mathbf{Q}_{\pm,t}^n = \frac{1}{4} \sum_{k=1}^{[nt]} \sqrt{a_k}\sqrt{a'_k} (\xi_k \pm \xi'_k)^2.$$

The proof relies on the following decomposition

$$\overline{\mathbf{C}}_t^n = \mathbf{Q}_{+,t}^n - \mathbf{Q}_{-,t}^n.$$

Proof of Proposition 2.1 By the independence of $\mathbf{Q}_{+,1}^n$ and $\mathbf{Q}_{-,1}^n$, we obtain that

$$\begin{aligned}\Lambda_n(\lambda) &= \frac{1}{n} \log \mathbb{E}(\exp(\lambda n \overline{\mathbf{C}}_1^n)) = \frac{1}{n} \log \mathbb{E}(\exp(\lambda n (\mathbf{Q}_{+,1}^n - \mathbf{Q}_{-,1}^n))) \\ &= \frac{1}{n} \log \mathbb{E}(\exp(\lambda n \mathbf{Q}_{+,1}^n)) + \frac{1}{n} \log \mathbb{E}(\exp(-\lambda n \mathbf{Q}_{-,1}^n)) := \Lambda_{n,+}(\lambda) + \Lambda_{n,-}(\lambda).\end{aligned}$$

Let us deal with $\Lambda_{n,+}$. We have that

$$\begin{aligned}\Lambda_{n,+}(\lambda) &= \frac{1}{n} \log \mathbb{E}(\exp(\lambda n \mathbf{Q}_{+,1}^n)) \\ &= \frac{1}{n} \sum_{k=1}^{n} P\left(n \frac{\lambda}{2} \left(\sqrt{\int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}^2 ds} \sqrt{\int_{t_{k-1}^n}^{t_k^n} \sigma_{2,s}^2 ds} + \int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}\sigma_{2,s}\rho_s ds \right) \right) \\ &= \int_0^1 P\left(\frac{\lambda}{2} f_n(t) \right) dt,\end{aligned}$$

where P is given in (2.3) and

$$f_n(t) := \sum_{k=1}^{n} 1_{(t_{k-1}^n, t_k^n]}(t) \frac{\sqrt{\int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}^2 ds} \sqrt{\int_{t_{k-1}^n}^{t_k^n} \sigma_{2,s}^2 ds} + \int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}\sigma_{2,s}\rho_s ds}{t_k^n - t_{k-1}^n}.$$

Let us remark that we have

$$f_n(t) = \sqrt{\sum_{k=1}^{n} 1_{(t_{k-1}^n, t_k^n]}(t) \frac{\int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}^2 ds \int_{t_{k-1}^n}^{t_k^n} \sigma_{2,s}^2 ds}{t_k^n - t_{k-1}^n}} + \sum_{k=1}^{n} 1_{(t_{k-1}^n, t_k^n]}(t) \frac{\int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s}\sigma_{2,s}\rho_s ds}{t_k^n - t_{k-1}^n}.$$

Clearly, $f_n(t)$ is a dt martingale w.r.t. the partially directed filtration $(\mathcal{B}_{\tau_n} := \sigma((t_{k-1}^n, t_k^n], k = 1, \cdots, n))_n$.

By the convexity of P and Jensen inequality, we obtain that

$$\int_0^1 P\left(\frac{\lambda}{2} f_n(t)\right) dt \leq \int_0^1 P\left(\frac{\lambda \sigma_{1,t}\sigma_{2,t}(1+\rho_t)}{2}\right) dt = \Lambda_+(\lambda).$$

On the other hand, by the classical Lebesgue derivation theorem, we have that

$$f_n(t) \longrightarrow f(t) := \sigma_{1,t}\sigma_{2,t} + \sigma_{1,t}\sigma_{2,t}\rho_t, \quad dt - a.e. \quad \text{on} \quad [0,1].$$

The continuity of $P : \mathbb{R} \to (-\infty, +\infty]$ gives

$$P\left(\frac{\lambda}{2} f_n(t)\right) \longrightarrow P\left(\frac{\lambda}{2} f(t)\right), \quad dt - a.e. \quad \text{on} \quad [0,1].$$

As $P\left(\frac{\lambda}{2} f_n(t)\right) \geq -\frac{|\lambda|}{2}\sigma_{1,t}\sigma_{2,t} \in L^1([0,1], dt)$, we can apply Fatou's lemma to conclude that

$$\liminf_{n\to\infty} \Lambda_{n,+}(\lambda) = \liminf_{n\to\infty} \int_0^1 P\left(\frac{\lambda}{2} f_n(t)\right) dt \geq \int_0^1 \liminf_{n\to\infty} P\left(\frac{\lambda}{2} f_n(t)\right) dt = \Lambda_+(\lambda).$$

Doing the same calculations with $\Lambda_{n,-}$, we obtain that

$$\Lambda_{n,-}(\lambda) \leq \int_0^1 P\left(-\frac{\lambda \sigma_{1,t}\sigma_{2,t}(1-\rho_t)}{2}\right) dt = \Lambda_-(\lambda),$$

and

$$\liminf_{n\to\infty} \Lambda_{n,-}(\lambda) \geq \Lambda_-(\lambda).$$

From below, we conclude that

$$\Lambda_n(\lambda) \leq \Lambda_+(\lambda) + \Lambda_-(\lambda) := \Lambda(\lambda),$$

and

$$\liminf_{n\to\infty} \Lambda_n(\lambda) \geq \Lambda(\lambda),$$

which implies that

$$\lim_{n\to\infty} \Lambda_n(\lambda) = \lim_{n\to\infty} (\Lambda_{n,+}(\lambda) + \Lambda_{n,-}(\lambda)) = \Lambda(\lambda).$$

Which ends the proof of first part of Proposition 2.1.

For the second part of Proposition 2.1, at first we will reduce the study to the case $u_l = 0$ for $l = 1, 2$. Let $\beta = \max(\| u_1 \|_\infty, \| u_2 \|_\infty)$. Since

$$\left|\mathbf{C}_1^n - \overline{\mathbf{C}_1^n}\right| \leq \frac{\beta^2}{n} + \frac{\beta}{n}\sum_{k=1}^n \left|\int_{t_{k-1}^n}^{t_k^n} \sigma_{1,s} dB_{1,s}\right| + \left|\int_{t_{k-1}^n}^{t_k^n} \sigma_{2,s} dB_{2,s}\right|.$$

For $l = 1, 2$, we have for all $\lambda > 0$ and all $\delta > 0$

$$\frac{1}{n}\log \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n \left|\int_{t_{k-1}^n}^{t_k^n} \sigma_{l,s} dB_{l,s}\right| \geq \delta\right) \leq -\delta\lambda + \frac{\lambda^2}{2n}\int_0^1 \sigma_{l,s}^2 ds.$$

Letting n go to infinity and then λ to infinity we get that for all $\delta > 0$

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n \left|\int_{t_{k-1}^n}^{t_k^n} \sigma_{l,s} dB_{l,s}\right| \geq \delta\right) = -\infty.$$

By the approximation technique (Theorem 4.2.13 in [3]), we deduce that $\mathbb{P}\left(\mathbf{C}_1^n \in \cdot\right)$ satisfies the same LDP as $\mathbb{P}\left(\overline{\mathbf{C}_1^n} \in \cdot\right)$. Hence we can assume that $u_l = 0$ for $l = 1, 2$.

Now, by inspection of the proof of Theorem 1.1 in [4], we deduce that the sequence $\mathbb{P}\left(\mathbf{Q}_{\pm,1}^n \in \cdot\right)$ satisfies the LDP on $\mathbb{R}$ with speed n and rate function given by

$$\Lambda_{\pm}^*(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda_{\pm}(\lambda)\}.$$

By the independence of the sequences $\mathbf{Q}_{+,1}^n$ and $\mathbf{Q}_{-,1}^n$, and the contraction principle, see Exercise 4.2.7 in [3], we deduce that $\mathbb{P}\left(\overline{\mathbf{C}}_1^n \in \cdot\right)$ satisfies the LDP with rate function

$$\Lambda^*(x) = \inf_{x = x_1 - x_2} \left\{\Lambda_+^*(x_1) + \Lambda_-^*(x_2)\right\}.$$

As we have also determined explicitly the logarithm of the moment generating function Λ, the rate function is also given by (2.1).

Proof of Theorem 2.2 The proof of the first part is very similar to Proposition 2.1. It is a consequence of Theorem 1.2 in [4] and the contraction principle. For the second part of Theorem 2.2, the same arguments give the large deviation with the rate function

$$I(\gamma) = \inf_{\gamma = \gamma_1 - \gamma_2} \left\{I_+(\gamma_1) + J_-(\gamma_2)\right\},$$

where

$$I_{\pm}(\gamma) = \int_0^1 P^* \left(\frac{2\dot{\gamma}(t)}{\sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t)}\right) dt + \int_0^1 \frac{1}{\sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t)} d\gamma^{\perp}.$$

An easy variational calculus gives the identification of the rate function in (2.4).

Proof of Theorem 2.3 As before, we treat only the case $u_l = 0$ for $l = 1, 2$. We have the following decomposition

$$\mathbf{C}_{\cdot}^n - \langle X_1, X_2 \rangle_{\cdot} = (\mathbf{Q}_{+,\cdot}^n - \alpha_{+,\cdot}) - (\mathbf{Q}_{-,\cdot}^n - \alpha_{-,\cdot}),$$

where the definition of $\alpha_{\pm}$ is given in (2.9).

Now using Theorem 1.3 in [4], we deduce that the sequence $\mathbb{P}\left(\frac{\sqrt{n}}{b_n}(\mathbf{Q}_{\pm,\cdot}^n - \alpha_{\pm,\cdot}) \in \cdot\right)$ satisfies the LDP on $\mathbb{D}_0([0,1])$ with speed b_n^2 and with the good rate function $J_{\pm,m}$ given by

$$J_{\pm,m}(\gamma) = \begin{cases} \displaystyle\int_0^1 \frac{\dot{\gamma}(t)^2}{\sigma_{1,t}^2\sigma_{2,t}^2(1 \pm \rho_t)^2} 1_{[t:\sigma_{1,t}\sigma_{2,t} > 0]} dt & \text{if } d\gamma \ll \sigma_{1,t}\sigma_{2,t}(1 \pm \rho_t)dt \\ +\infty & \text{otherwise.} \end{cases}$$

By the same argument as before, we deduce that $\mathbb{P}\left(\frac{\sqrt{n}}{b_n}(\mathbf{C}_{\cdot}^n - \langle X_1, X_2 \rangle_{\cdot}) \in \cdot\right)$ satisfies the LDP on $\mathbb{D}_0([0,1])$ with speed b_n^2 and with the good rate function J_m given by

$$J_m(\gamma) = \inf_{\gamma = \gamma_1 - \gamma_2} \left\{J_{+,m}(\gamma_1) + J_{-,m}(\gamma_2)\right\}.$$

An easy calculation gives the identification of the rate function in (2.6).

Acknowledgments: The authors want to thank Professor Liming Wu for fruitful discussions. We are grateful to the anonymous referees for the valuable comments.

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